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A New Computation Technique of Glauert's Integrals

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Abstract—This paper deals with the study of particular functions of a complex variable which enable the easy calculation of Glauert's integrals.

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In aerodynamics, in order to apply the "Thin Airfoils Theory" by Glauert, the so-called Glauert's integrals are to be solved:

$$G_n = \int_0^{\pi} \frac{\cos(n\theta')}{\cos\theta' - \cos\theta}\,\mathrm{d}\theta' = \pi\,\frac{\sin(n\theta)}{\sin\theta}, \tag{1}$$

$$G_n^* = \int_0^{\pi} \frac{\sin(n\theta')\sin\theta'}{\cos\theta' - \cos\theta}\,\mathrm{d}\theta' = -\pi\,\cos(n\theta), \tag{2}$$

with $0 \le \theta \le \pi$. Glauert's integral (1) can be solved, for example, integrating in the complex plane the function

$$f(z) = \frac{z^n}{z^2 - 2z\cos\theta + 1}$$

with respect to z around the circle of unit radius $z = \mathrm{e}^{\mathrm{i}\theta'}$ and then equating the imaginary parts [1]. The integral (2) can be reduced to the form (1) by putting $\sin(n\theta')\sin\theta' = (1/2)[\cos(n-1)\theta' - \cos(n+1)\theta']$.

The solutions of Glauert's integrals can also be easily obtained, almost as a by-product, applying a new method that will be shortly described here.

Suppose we have on the strip $[(-c/2),(c/2)]$ on the x-axis a two-dimensional source distribution of strength $q(x)$ per unit length (c could be the chord of a symmetric airfoil). With $z = x + \mathrm{i}y$, the complex velocity at any point of the complex plane is given by

$$u - \mathrm{i}v = \frac{1}{2\pi}\int_{-c/2}^{c/2} \frac{q(x')}{z - x'}\,\mathrm{d}x'. \tag{3}$$

Let us consider the behaviour of the complex velocity $u_0 - \mathrm{i}v_0$ on the x-axis. The real part $u_0(x)$ is even on the whole axis, i.e., it does not change its sign through the axis, and vanishes at infinite distance from the strip

$$u_0(x) = \frac{1}{2\pi}\int_{-c/2}^{c/2} \frac{q(x')}{x - x'}\,\mathrm{d}x'. \tag{4}$$

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The imaginary part $-v_0(x)$, equal to zero outside the strip $[(-c/2),(c/2)]$, is odd on the strip, i.e., its sign changes through the strip, which must be treated as a double-faced strip

$$v_0(x) = \pm\frac{q(x)}{2}, \tag{5}$$

where the signs + and − are valid, respectively, on the upper and the lower face of the strip.

The properties exhibited by the velocity components on the x-axis due to a source distribution on a strip of the same axis are a particular case of a more general property of the functions of a complex variable.

Suppose that we want to find the function of a complex variable $F(z) = f + \mathrm{i}g$ in the z-plane, assuming that the real part is even on the x-axis and equal to zero at infinity, and that the imaginary part is equal to zero on the x-axis except on the strip $[(-c/2),(c/2)]$, where it is odd and known. If $g_{0+}(x)$ is the value of the imaginary part on the upper surface of the double strip, according to equations (3)–(5) we obtain:

$$F(z) = f + \mathrm{i}g = -\frac{1}{\pi}\int_{-c/2}^{c/2} \frac{g_{0+}(x')}{z - x'}\,\mathrm{d}x', \tag{6}$$

$$f_0(x) = -\frac{1}{\pi}\int_{-c/2}^{c/2} \frac{g_{0+}(x')}{x - x'}\,\mathrm{d}x'. \tag{7}$$

The above properties have been successfully applied to determine the complex velocity and the pressure distribution around a thin airfoil in unsteady incompressible flow [2].

To generalize this method to evaluate Glauert's integrals, now we can examine three particular functions which lead rapidly to the required solutions without any actual integration. Suppose that the variable $z = x + \mathrm{i}y$ is obtained from the variable $\overline{z}$ by means of the Zhukowsky transformation

$$z = \overline{z} + \frac{c^2}{16\overline{z}}. \tag{8}$$

If $\overline{z} = (c/4)\mathrm{e}^{\mathrm{i}\theta}$, equation (8) yields $z = (c/2)\cos\theta$. While $\overline{z}$ varies on the circle of radius $c/4$, z varies on the double strip $[(-c/2),(c/2)]$; the upper face of the strip corresponds to $0 \le \theta \le \pi$ and the lower one to $-\pi \le \theta \le 0$.

On the double strip, we have

$$x = \frac{c}{2}\cos\theta; \qquad \mathrm{d}x = -\frac{c}{2}\sin\theta\,\mathrm{d}\theta; \qquad \int_{-c/2}^{c/2} \cdots \mathrm{d}x' = \int_0^{\pi} \cdots \frac{c}{2}\sin\theta'\,\mathrm{d}\theta'. \tag{9}$$

Consider first the function

$$F(z) = f + \mathrm{i}g = \frac{c/2}{\sqrt{z^2 - c^2/4}}, \tag{10}$$

that satisfies the conditions for the validity of equations (6)–(7). On the double strip, the real part is zero and the imaginary part is an odd function

$$f_0 = 0; \qquad g_{0+} = -\frac{1}{\sin\theta}. \tag{11}$$

Equations (6), (7), (9), (11) yield

$$F(z) = f + \mathrm{i}g = \frac{1}{\pi}\int_0^{\pi} \frac{\mathrm{d}\theta'}{2(z/c) - \cos\theta'}, \tag{12}$$

$$f_0 = 0 = \frac{1}{\pi}\int_0^{\pi} \frac{\mathrm{d}\theta'}{\cos\theta - \cos\theta'}. \tag{13}$$

So, we have obtained the solution of integral (1) with $n = 0$: $G_0 = 0$.

Consider now the function

$$F(z) = f + \mathrm{i}g = \frac{1}{((4/c)\overline{z})^n} = \cos(n\theta) - \mathrm{i}\sin(n\theta). \tag{14}$$

On the double strip, we obtain

$$f_0 = \cos(n\theta); \qquad g_{0+} = -\sin(n\theta). \tag{15}$$

Equations (6), (7), (9), (15) yield

$$F(z) = f + \mathrm{i}g = \frac{1}{\pi}\int_0^{\pi} \frac{\sin(n\theta')\,\sin\theta'}{2(z/c) - \cos\theta'}\,\mathrm{d}\theta', \tag{16}$$

$$f_0 = \cos(n\theta) = \frac{1}{\pi}\int_0^{\pi} \frac{\sin(n\theta')\,\sin\theta'}{\cos\theta - \cos\theta'}\mathrm{d}\theta'. \tag{17}$$

Equation (17) gives the solution of integral (2) G_n^*.

Finally consider the function

$$F(z) = f + \mathrm{i}g = \frac{1}{((4/c)\overline{z})^n}\,\frac{c/2}{\sqrt{z^2 - c^2/4}}. \tag{18}$$

Its value on the double strip is

$$[\cos(n\theta) - \mathrm{i}\sin(n\theta)]\left(-\frac{\mathrm{i}}{\sin\theta}\right),$$

and then we obtain

$$f_0 = -\frac{\sin(n\theta)}{\sin\theta}; \qquad g_{0+} = -\frac{\cos(n\theta)}{\sin\theta}. \tag{19}$$

Equations (6), (7), (9), (19) give

$$F(z) = f + \mathrm{i}g = \frac{1}{\pi}\int_0^{\pi} \frac{\cos(n\theta')}{2(z/c) - \cos\theta'}\,\mathrm{d}\theta', \tag{20}$$

$$f_0 = -\frac{\sin(n\theta)}{\sin\theta} = \frac{1}{\pi}\int_0^{\pi} \frac{\cos(n\theta')}{\cos\theta - \cos\theta'}\,\mathrm{d}\theta'. \tag{21}$$

From equation (21), we obtain the solution of integral (1) G_n.

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